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EXACT ILYUSHIN YIELD SURFACE

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Abstract—The approximate Ilyushin yield surface is widely used in structural calculations to represent full plasticity in stress-resultant space. The exact version of the surface has not been used, because the parametric form in which it was described by Ilyushin was not amenable to calculation. This paper presents a reparametrization of the Ilyushin yield criterion for thin plates which produces a simpler (though still exact) form which is suitable for use in practical formulations. A method is outlined by which the exact Ilyushin yield criterion can be used in such calculations, without the need to use approximate yield functions. A method for calculating positions on the surface corresponding to a linear multiple of any set of stress-resultants is presented. Various features of the exact yield surface are described.

NOMENCLATURE

e_x, e_y, e_{xy}	strains
f	yield function in stress space (von Mises')
m_x, m_y, m_{xy}	non-dimensional flexural stress resultant components
n_x, n_y, n_{xy}	non-dimensional in-plane stress resultant components
z	non-dimensional position through thickness of plate
C, C_1, C_2	arbitrary constants
F	yield function in stress resultant space
F_t, F_{tm}, F_m	components of normal to yield surface
J_0, J_1, J_2	through-thickness integrals [eqn (20)]
K_0, K_1, K_2	through-thickness integrals [eqns (25) and (30)]
$P_\varepsilon, P_{\varepsilon\kappa}, P_\kappa$	quadratic strain intensities
Q_1, Q_{12}, Q_2	quadratic stress resultants (not on the yield surface)
Q_t, Q_{tm}, Q_m	quadratic stress resultant, on the yield surface
S	Young's modulus
α, β, γ	fundamental parameter of the present formulation
$\alpha_0, \beta_0, \gamma_0$	initial estimates of α, β and γ
$\varepsilon_x, \varepsilon_y, \varepsilon_{xy}$	mid-plane strain components
$\kappa_x, \kappa_y, \kappa_{xy}$	non-dimensional curvature components
ζ, μ	fundamental parameter of Ilyushin's formulation
λ	plastic strain rate multiplier
η	multiplier relating given set of Q_i to the yield surface [eqn (45)]
η_0	initial estimate of η [eqn (51)]
$\psi, \phi, \chi, \Delta, \Delta_1$	derived parameter of Ilyushin's formulation [eqn (8)]
$\sigma_x, \sigma_y, \sigma_{xy}$	stresses (dimensional)
σ_0	uniaxial yield stress.

INTRODUCTION

For many years, research has been carried out to determine the collapse loads and collapse modes of steel-plated structures, spurred on by the failure of box girder bridges and major developments in the offshore industry. More recently, studies of the behaviour of aircraft and motor vehicles in crash simulations have also been carried out.

Because of the necessity of producing results for these problems as quickly as possible, some simplifications were made in the analyses. These approximations were assumed to have little effect on the theoretical solutions, and good correlations have been found between theory and experiment.

However, it is not always necessary to make all of these simplifications. This paper shows how the Ilyushin Yield Surface can be used in practical calculations without the need to introduce the usual approximations.

SINGLE LAYER APPROXIMATIONS

In order to determine the maximum collapse load of a plate, a criterion is needed to assess when the plate reaches a situation where the behaviour is governed by plasticity. Two approaches have been identified. It is possible to work either in terms of stresses which vary through the thickness of the plate (Moxham, 1971; Little, 1977; Harding *et al.*, 1977), in which case a yield criterion such as von Mises' is used, or in terms of stress resultants (Crisfield, 1973; Frieze, 1975), when a more complex full plasticity yield surface is needed. The arguments for doing one or the other have been rehearsed elsewhere (Crisfield, 1973; Bradfield, 1982), but may be summarized briefly as a trade-off between accuracy at the expense of large computer resources when using a stress formulation, and reduced computation but less accuracy when dealing with stress resultants. Both approaches are useful, and each can be worthwhile in different circumstances.

When dealing with stress resultants, it is, of course, important to be able to identify a yield surface, which marks the limiting values of the stress resultants, beyond which the plate may not be loaded. The vast majority of the work carried out in this field (Bieniek and Funaro, 1976; Eggers and Kroplin, 1978; Dinis and Owen, 1982; Trueb, 1983) has utilized what was usually referred to as the "Ilyushin yield surface", although in reality, the yield surface was only an approximation to the exact surface derived by Ilyushin.

Ilyushin (1948) derived an exact form of the yield surface for a linear elastic, perfectly plastic isotropic material which obeys von Mises' yield criterion; this will be referred to as the "Exact Ilyushin Yield Surface" to distinguish it from the approximate form mentioned above. Given the complete absence of computing power at the time, it was of purely academic interest.

Ilyushin published his work in French (Ilyushin, 1956), but an English version of his work was not available until it was translated by Crisfield (1974). The notation used in this paper will follow, where appropriate, the notation of Crisfield's translation, in common with most references to this subject.

Consider a plate of thickness t. The through thickness coordinates are measured from the mid-surface by a dimension (s), although it will be more convenient to define a non-dimensional coordinate z $(=s/t)$. In-plane stresses $(\sigma_x, \sigma_y, \sigma_{xy})$ occur through the plate, and non-dimensional stress resultants are defined by:

$$n_x = \frac{1}{\sigma_0}\int_{-1/2}^{1/2} \sigma_x \, dz \quad \text{and} \quad m_x = \frac{4}{\sigma_0}\int_{-1/2}^{1/2} \sigma_x \, dz, \tag{1}$$

with similar expressions for n_y, n_{xy}, m_y and m_{xy}.

The material is assumed to obey von Mises' yield criterion:

$$f = \frac{(\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\sigma_{xy}^2)}{\sigma_0^2} = 1. \tag{2}$$

It will then be convenient to define quadratic stress intensities Q_t, Q_m and Q_{tm} as:

$$\begin{aligned} Q_t &= n_x^2 + n_y^2 - n_x n_y + 3n_{xy}^2, \\ Q_{tm} &= m_x n_x + m_y n_y - \tfrac{1}{2}(m_x n_y + m_y n_x) + 3m_{xy}n_{xy}, \\ Q_m &= m_x^2 + m_y^2 - m_x m_y + 3m_{xy}^2. \end{aligned} \tag{3}$$

Non-dimensionalized mid-plane strains $(\varepsilon_x, \varepsilon_y, \varepsilon_{xy})$ and curvatures $(\kappa_x, \kappa_y, \kappa_{xy})$ are defined (and will be referred to as strain resultants), and there will be corresponding non-dimensional strains at each level defined by

$$e_x = \varepsilon_x + 4z\kappa_x, \quad \text{etc.} \tag{4}$$

If the plate is elastic at depth z, the σ_x stress at that depth is given by

$$\frac{(\sigma_x)_z}{\sigma_0} = e_x + \nu e_y, \tag{5}$$

with similar expressions for σ_y and σ_{xy}. (The true "engineering" strains and curvatures are given by $\varepsilon_x\sigma_0(1-\nu^2)/S$ and $\kappa_x 4\sigma_0(1-\nu^2)/(St)$ respectively, where S is Young's modulus.)

Corresponding to the quadratic stress intensities, quadratic strain intensities can also be defined:

$$\begin{aligned} P_\varepsilon &= d\varepsilon_x^2 + d\varepsilon_y^2 + d\varepsilon_x\, d\varepsilon_y + 0.25\, d\varepsilon_{xy}^2, \\ P_{\varepsilon\kappa} &= 4(d\varepsilon_x\, d\kappa_x + \tfrac{1}{2}(d\varepsilon_x\, d\kappa_y + d\varepsilon_y\, d\kappa_x) + d\varepsilon_y\, d\kappa_y + 0.25\, d\varepsilon_{xy}\, d\kappa_{xy}), \\ P_\kappa &= 16(d\kappa_x^2 + d\kappa_y^2 + d\kappa_x\, d\kappa_y + 0.25\, d\kappa_{xy}^2). \end{aligned} \tag{6}$$

ILYUSHIN'S DERIVATION

There are six stress resultants for any element of the plate, n_x, n_y, n_{xy}, m_x, m_y and m_{xy}, so the yield surface will be a function of five parameters. By considering the three non-dimensional quadratic stress intensities, Q_t, Q_m and Q_{tm}, the surface can be reduced to a surface in a 3-dimensional space, and can thus be represented by two independent parameters. Ilyushin (1948) represented the surface in terms of two non-dimensional parameters ζ and μ [the physical significance of which will be given later in eqn (22)]. The derivation can be found in Crisfield (1974), and the resulting equations summarized as:

$$\begin{aligned} Q_t &= \frac{1}{\Delta_1^2}(\mu^2\psi^2 + \phi^2), \\ Q_{tm} &= \frac{2}{\Delta_1^3}(\mu^2\Delta\psi^2 + \Delta\phi^2 + \mu^2\phi\psi + \phi\chi), \\ Q_m &= \frac{4}{\Delta_1^4}(\mu^2\psi^2(\mu^2+\Delta^2) + \phi^2(4\mu^2+\Delta^2) + 2\mu^2\Delta\phi\psi - 2\mu^2\psi\chi + 2\Delta\phi\chi + \chi^2), \end{aligned} \tag{7}$$

where

$$\begin{aligned} \phi &= \zeta - 1, \\ \psi &= \left| \log_e \frac{(1+\sqrt{1-\mu^2})}{\mu} \pm \log_e \frac{(\zeta + \sqrt{\zeta^2-\mu^2})}{\mu} \right|, \\ \chi &= \left| \sqrt{1-\mu^2} \pm \zeta\sqrt{\zeta^2-\mu^2} \right|, \\ \Delta_1 &= \sqrt{1-\mu^2} \pm \sqrt{\zeta^2-\mu^2}, \\ \Delta &= \frac{1-\zeta^2}{\Delta_1}, \end{aligned} \tag{8}$$

subject to the conditions that

$$0 \leqslant \mu \leqslant 1,$$
$$\mu \leqslant \zeta \leqslant 1. \tag{9}$$

(Note that Ilyushin actually used (λ) instead of (ζ) written here; this has been changed to avoid confusion with the plastic strain rate multiplier which is conventionally also represented by λ, and which will be needed below.)

The surface is bounded by the condition that

$$Q_t Q_m \geqslant Q_{tm}^2, \tag{10}$$

which corresponds to $\mu = 0$. This is derived from the Schwarz inequality which states that

$$|\mathbf{a}| \cdot |\mathbf{b}| \geqslant |\mathbf{a} \cdot \mathbf{b}|, \tag{11}$$

where **a** and **b** are vectors. If **a** is taken as $(n_x - 0.5n_y, \sqrt{3}/2n_y, \sqrt{3}n_{xy})$, and **b** taken as the corresponding moment terms, inequality (10) results.

Figures 1 and 2 show perspective views of the surface, plotted in Q_t, Q_m, Q_{tm} space, with lines of constant Q_t, Q_m and Q_{tm} shown. There is a discontinuity in slope which is shown more clearly in Fig. 2. The plot is symmetrical about the plane $Q_{tm} = 0$, and could also be plotted without loss of clarity as Q_{tm} against $Q_t - Q_m$, since there are no folds of the surface or hidden views in such a plot. One half of the surface will be shown in this way in some of the subsequent figures.

In his original paper, Ilyushin proposed an approximation to his exact surface, and it is this approximation to which most authors refer when discussing the Ilyushin Yield Surface:

$$Q_t + \frac{1}{\sqrt{3}}|Q_{tm}| + Q_m = 1. \tag{12}$$

This is a very crude approximation (Fig. 3), consisting simply of two planes in Q-space. The discontinuity this introduces at the line of symmetry $Q_{tm} = 0$ has been a major cause of problems when using the surface, not so much for the error in the position of the surface, but because it produces a discontinuity in the normal to the surface and consequently also in the plate rigidities calculated from the normal direction.

Various other approximations have been made to the exact Ilyushin surface, and these are discussed in some detail by Robinson (1971). This work was discussed by one of the

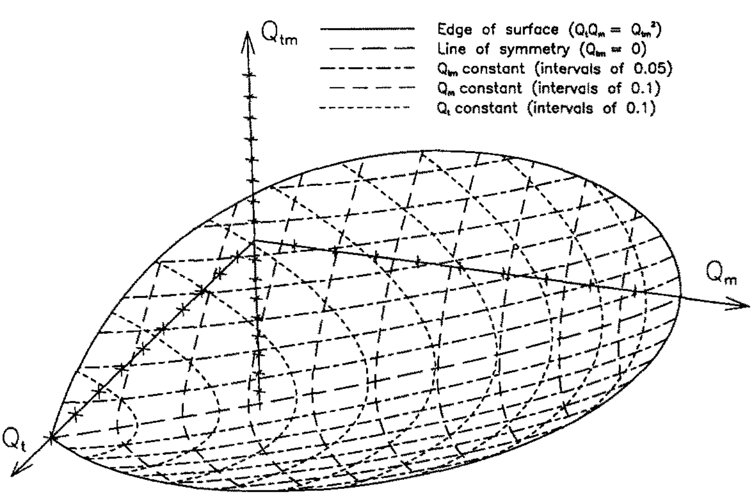

Fig. 1. Three-dimensional view of exact Ilyushin Yield Surface.

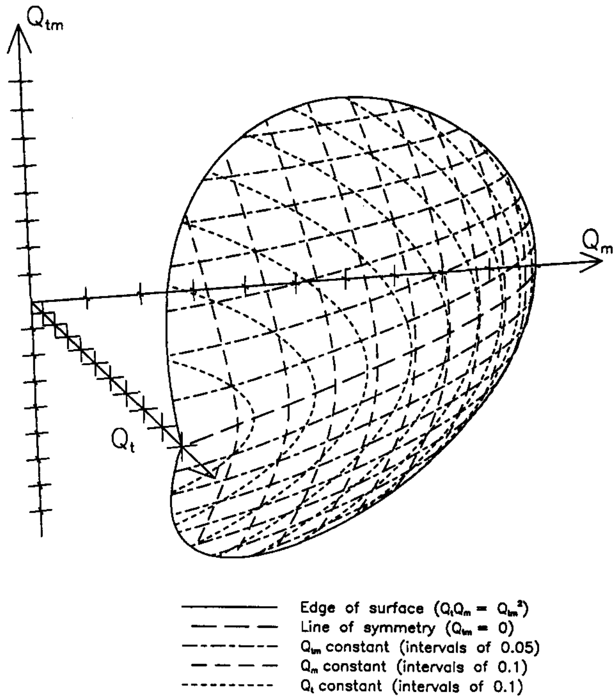

Fig. 2. Three-dimensional view of surface showing slope discontinuity.

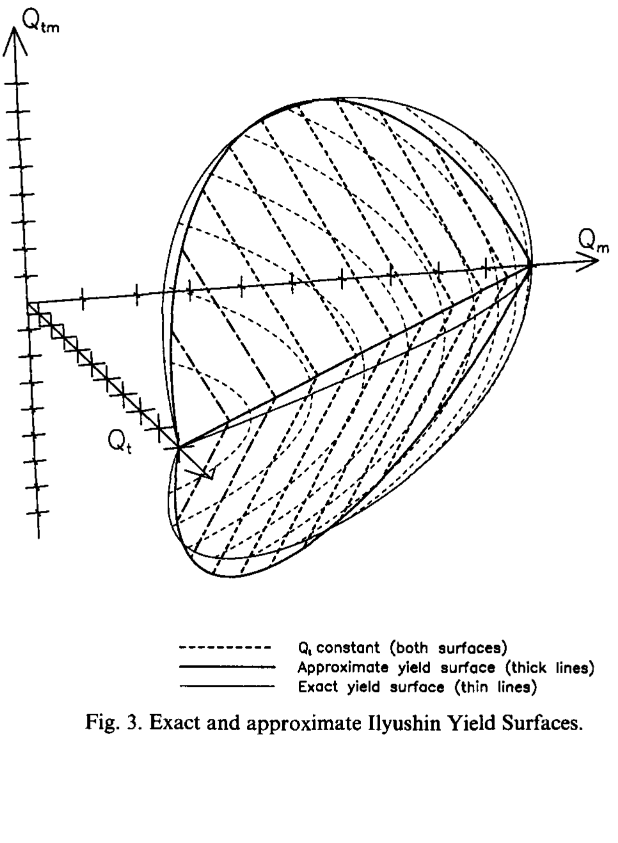

Fig. 3. Exact and approximate Ilyushin Yield Surfaces.

present authors (Burgoyne, 1979), and one of the alternative yield surfaces, that due to Ivanov (1967), has been used subsequently in numerical studies of plate buckling problems (Crisfield, 1979). Ivanov uses a quadratic representation of the yield surface in Q-space, as opposed to the linear representation of the approximate Ilyushin yield surface:

$$Q_t + Q_m/2 + \sqrt{Q_m^2/4 + Q_{tm}^2} - \frac{1}{4}\left(\frac{Q_t Q_m - Q_{tm}^2}{Q_t + 0.48 Q_m}\right) = 1. \tag{13}$$

The last term on the left-hand side is sometimes omitted, giving a less accurate approximation but one which is easier to use in certain circumstances. Ivanov's surface overcomes many of the difficulties associated with the approximate Ilyushin yield surface; it has no discontinuities except one in slope at $Q_t = 1$ where the exact surface also has a slope discontinuity, and always lies within 1% of the exact surface. Nonetheless, buckling problems are sensitive to changes in the elasto-plastic rigidities, which are, of course, derived from the normal to the surface. Thus, any error in location of the surface will be reflected in a larger error in the normal, and hence in the rigidities. It is thus worthwhile considering whether the exact surface itself can be used as the basis of a more accurate calculation.

REPARAMETRIZATION OF THE YIELD SURFACE

Ilyushin's choice of the two independent parameters ζ and μ makes it necessary to divide the surface into four regions before a solution can be obtained. The surface is governed by different equations on each side of the plane of symmetry, and within each half of the surface two areas have to be identified, known as the "in-plane dominant" and "bending dominant" regions. [The different equations arise by taking the alternative signs for the $\pm$ terms in eqns (8).] Figure 4 shows lines of constant ζ and μ for one half of the surface, and Fig. 5 shows a detail of one of the "in-plane dominant" regions. It is clear that even without the problem of the different regions, a solution based on ζ and μ would not form a suitable basis for use in a plate analysis program, since lines of constant ζ are virtually parallel to lines of constant μ in many cases. In these regions, equations set up in terms of ζ and μ would be ill-conditioned and numerically unstable.

ζ and μ are not the only two parameters that could be used. It is feasible to derive expressions to define the surface in terms of other parameters, following a derivation that is similar to Ilyushin's.

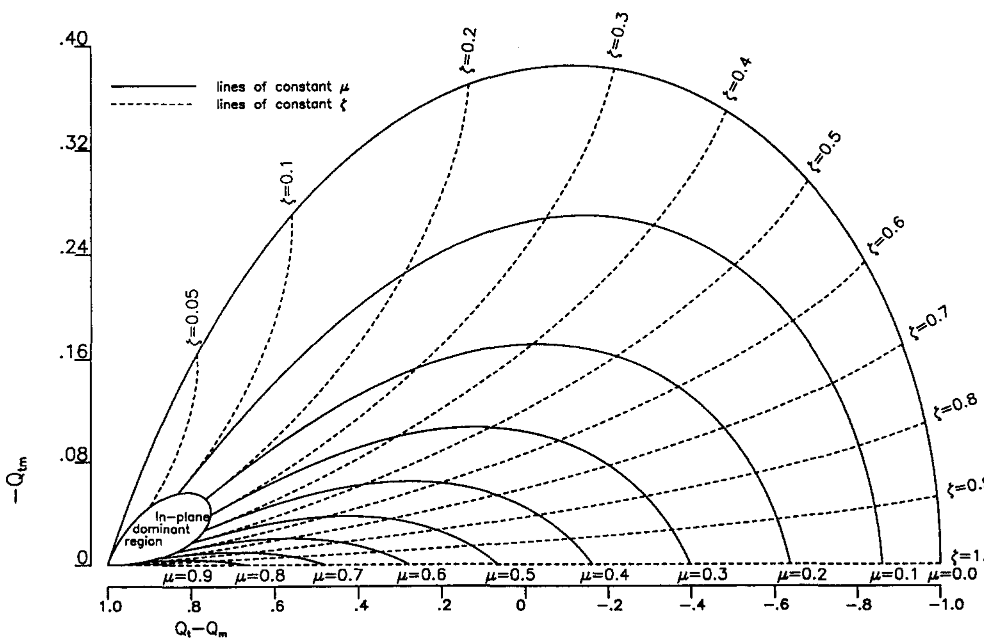

Fig. 4. Parametric plot showing Ilyushin's original parameters ζ, μ.

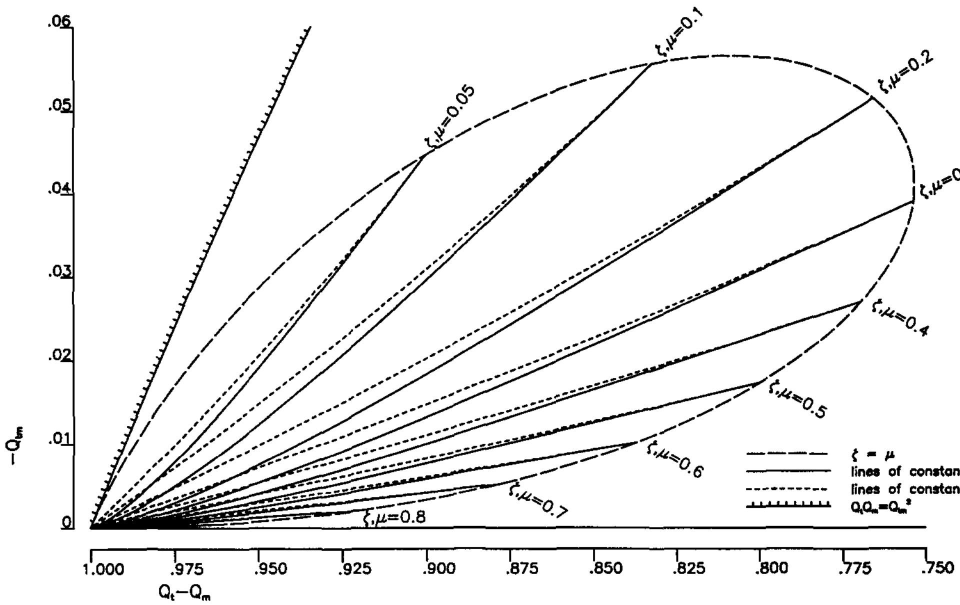

Fig. 5. In-plane dominant region constructed in terms of Ilyushin's original parameters ζ, μ.

Consider incremental strain resultants $d\varepsilon_x$, $d\varepsilon_y$ and $d\varepsilon_{xy}$, and $d\kappa_x$, $d\kappa_y$ and $d\kappa_{xy}$. In the limit, these strain increments will be wholly plastic, and thus normal to the von Mises stress yield surface, fixing the position on the yield surface and hence the stresses at any depth within the plate. This assumption about the relative magnitudes of the elastic and plastic strain components will be significant when discussing the normality law in a companion paper (Burgoyne and Brennan, 1993).

The strain increment vector is defined by the normality law in stress space

$$de_x = d\lambda \frac{\partial f}{\partial \sigma_x}, \quad de_y = d\lambda \frac{\partial f}{\partial \sigma_y}, \quad de_{xy} = d\lambda \frac{\partial f}{\partial \sigma_{xy}}, \tag{14}$$

where $d\lambda$ is the plastic strain rate multiplier. Then,

$$\begin{bmatrix} de_x \\ de_y \\ de_{xy} \end{bmatrix} = \frac{d\lambda}{\sigma_0} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}, \tag{15}$$

which may be readily inverted to give

$$\frac{1}{\sigma_0} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} = \frac{1}{3d\lambda} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} de_x \\ de_y \\ de_{xy} \end{bmatrix}. \tag{16}$$

The unknown multiplier $d\lambda$ can be obtained by substituting the above stress terms into the von Mises equation (2) to give

$$d\lambda = \left(\sqrt{de_x^2 + de_x\, de_y + de_y^2 + \tfrac{1}{4} de_{xy}^2}\right) / \sqrt{3}, \tag{17}$$

which can also be written as

$$d\lambda = (\sqrt{P_\varepsilon + 2zP_{\varepsilon\kappa} + z^2 P_\kappa})/\sqrt{3}. \tag{18}$$

The stresses may then be integrated over the plate thickness to get the stress resultants, giving

$$\begin{bmatrix} n_x \\ n_y \\ n_{xy} \\ m_x \\ m_y \\ m_{xy} \end{bmatrix} = \begin{bmatrix} 2J_0 & J_0 & 0 & 8J_1 & 4J_1 & 0 \\ J_0 & 2J_0 & 0 & 4J_1 & 8J_1 & 0 \\ 0 & 0 & J_0/2 & 0 & 0 & 2J_1 \\ 8J_1 & 4J_1 & 0 & 32J_2 & 16J_2 & 0 \\ 4J_1 & 8J_1 & 0 & 16J_2 & 32J_2 & 0 \\ 0 & 0 & 2J_1 & 0 & 0 & 8J_2 \end{bmatrix} \begin{bmatrix} d\varepsilon_x \\ d\varepsilon_y \\ d\varepsilon_{xy} \\ d\kappa_x \\ d\kappa_y \\ d\kappa_{xy} \end{bmatrix}, \tag{19}$$

where the J_i are the integrals

$$J_i = \frac{1}{\sqrt{3}} \int_{-1/2}^{1/2} \frac{z^i}{\sqrt{P_\varepsilon + 2P_{\varepsilon\kappa} z + P_\kappa z^2}} \cdot dz. \tag{20}$$

This set of six equations can be reduced to three by using quadratic forms of the stress and strain quantities:

$$\begin{bmatrix} Q_t \\ Q_{tm}/4 \\ Q_m/16 \end{bmatrix} = 3 \begin{bmatrix} J_0^2 & J_0 J_1 & J_1^2 \\ J_0 J_1 & (J_0 J_2 + J_1^2)/2 & J_1 J_2 \\ J_1^2 & J_1 J_2 & J_2^2 \end{bmatrix} \begin{bmatrix} P_\varepsilon \\ 2P_{\varepsilon\kappa} \\ P_\kappa \end{bmatrix}. \tag{21}$$

At this point, the present analysis diverges from that of Ilyushin, since he introduced the non-dimensional parameters:

$$\zeta = \left(\frac{P_\varepsilon - P_{\varepsilon\kappa} + P_\kappa/4}{P_\varepsilon + P_{\varepsilon\kappa} + P_\kappa/4}\right)^{1/2},$$

$$\mu = \left(\frac{P_\varepsilon P_\kappa - P_{\varepsilon\kappa}^2}{P_\kappa (P_\varepsilon + P_{\varepsilon\kappa} + P_\kappa/4)}\right)^{1/2}. \tag{22}$$

Instead, the parameters

$$\alpha = \frac{P_\varepsilon}{P_\kappa},$$

$$\beta = -\frac{P_{\varepsilon\kappa}}{P_\kappa}, \tag{23}$$

are defined. [It is convenient to use the negative sign in the definition of β, since β then has the physical meaning of being the position within the thickness of the plate where the equivalent strain rate increment, $d\lambda$ in eqns (17) and (18), is a minimum.]

With these substitutions, the stress intensities can be expressed as

$$\begin{bmatrix} Q_t \\ Q_{tm}/4 \\ Q_m/16 \end{bmatrix} = \begin{bmatrix} K_0^2 & K_0 K_1 & K_1^2 \\ K_0 K_1 & (K_0 K_2 + K_1^2)/2 & K_1 K_2 \\ K_1^2 & K_1 K_2 & K_2^2 \end{bmatrix} \begin{bmatrix} \alpha \\ -2\beta \\ 1 \end{bmatrix}, \tag{24}$$

where

$$K_i = \int_{-1/2}^{1/2} \frac{z^i}{\sqrt{\alpha - 2\beta z + z^2}} \cdot \mathrm{d}z = \sqrt{3P_\kappa} \cdot J_i. \tag{25}$$

The integrals can be performed analytically, giving

$$\begin{aligned} K_0 &= \log_e \left| \frac{\sqrt{\alpha - \beta + 0.25} + (0.5 - \beta)}{\sqrt{\alpha + \beta + 0.25} - (0.5 + \beta)} \right|, \\ K_1 &= \sqrt{\alpha - \beta + 0.25} - \sqrt{\alpha + \beta + 0.25} + \beta K_0, \\ 2K_2 &= (0.5 + \beta)\sqrt{\alpha - \beta + 0.25} + (0.5 - \beta)\sqrt{\alpha + \beta + 0.25} + 2\beta K_1 - (\alpha - \beta^2) K_0. \end{aligned} \tag{26}$$

Ilyushin's parameters may be derived from those used here by the substitutions:

$$\begin{aligned} \zeta &= \left(\frac{\alpha - \beta + 0.25}{\alpha + \beta + 0.25} \right)^{1/2}, \\ \mu &= \left(\frac{\alpha - \beta^2}{\alpha + \beta + 0.25} \right)^{1/2}. \end{aligned} \tag{27}$$

Figure 6 shows the yield surface with lines of constant α and lines of constant β.

In practice, it will also be useful to refer to a third parameter γ, which is not independent of α and β, but is given by

$$\gamma = \alpha - \beta^2. \tag{28}$$

Equations (24) can then be rewritten:

$$\begin{aligned} Q_t &= (\beta K_0 - K_1)^2 + \gamma K_0^2, \\ Q_{tm} &= 4(\beta K_0 - K_1)(\beta K_1 - K_2) + 4\gamma K_0 K_1, \\ Q_m &= 16(\beta K_1 - K_2)^2 + 16\gamma K_1^2, \end{aligned} \tag{29}$$

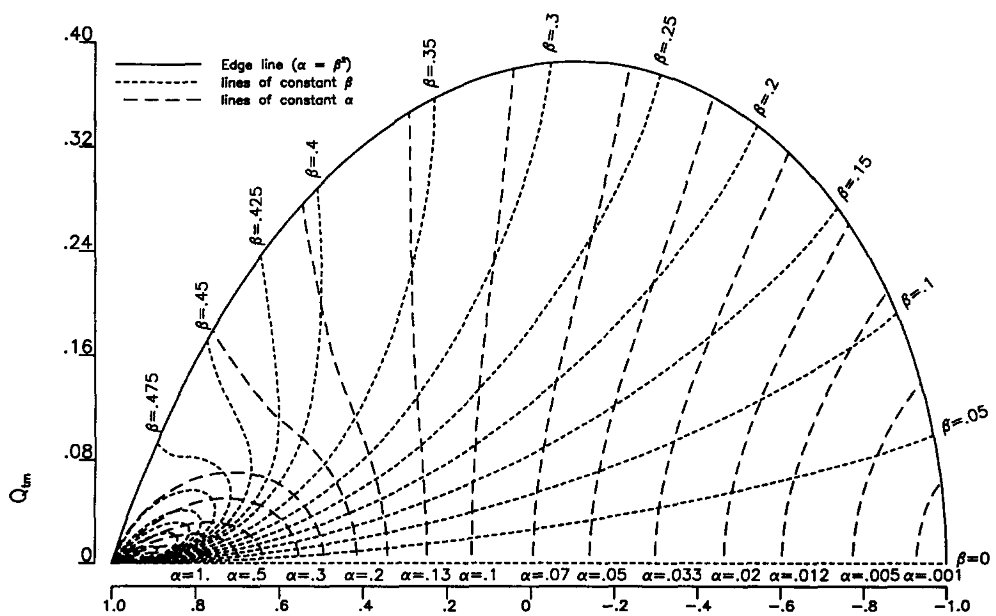

Fig. 6. Yield surface constructed in terms of α and β.

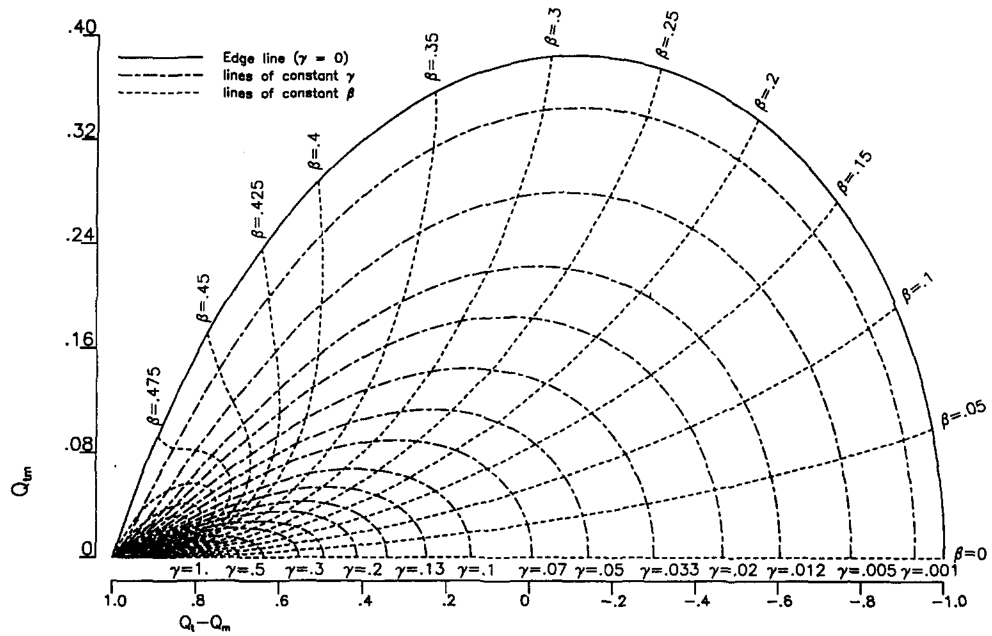

Fig. 7. Yield surface constructed in terms of β and γ.

and eqns (26) become

$$K_0 = \log_e \left| \frac{\sqrt{(0.5-\beta)^2+\gamma}+(0.5-\beta)}{\sqrt{(0.5+\beta)^2+\gamma}-(0.5+\beta)} \right|,$$

$$K_1 = \sqrt{(0.5-\beta)^2+\gamma}-\sqrt{(0.5+\beta)^2+\gamma}+\beta K_0,$$

$$2K_2 = (0.5+\beta)\sqrt{(0.5-\beta)^2+\gamma}+(0.5-\beta)\sqrt{(0.5+\beta)^2+\gamma}+2\beta K_1-\gamma K_0, \qquad (30)$$

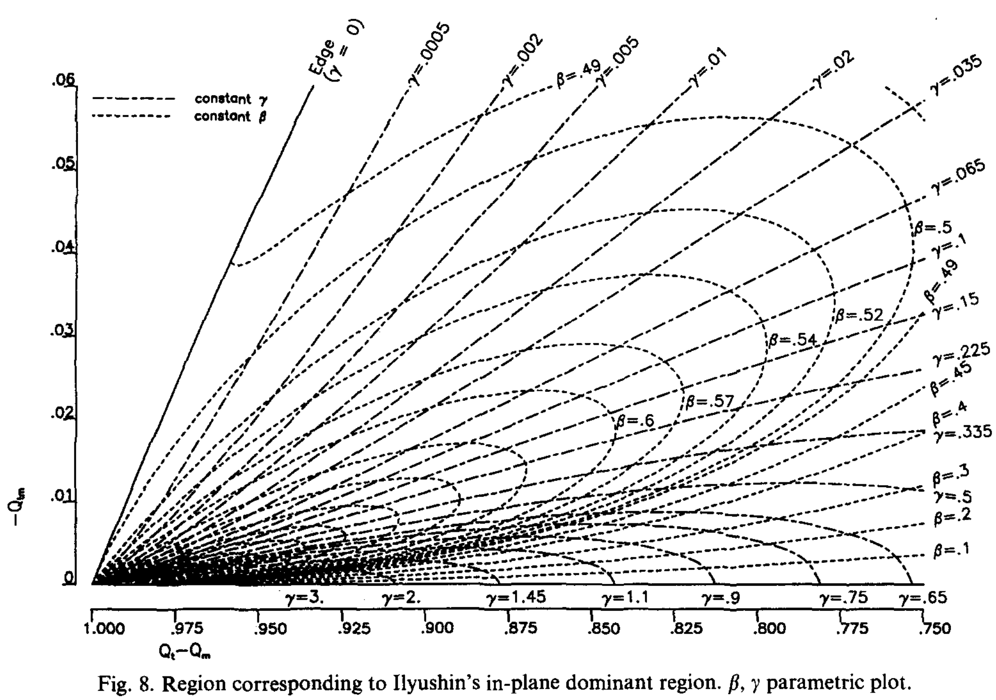

Fig. 8. Region corresponding to Ilyushin's in-plane dominant region. β, γ parametric plot.

subject to the limits

$$0 \leqslant \beta^2 \leqslant \alpha \leqslant \infty,$$
$$0 \leqslant \gamma \leqslant \infty. \tag{31}$$

Figure 7 shows lines of constant β and γ, while Fig. 8 shows a detail of the region for high Q_t (corresponding to the "in-plane dominant region" in Ilyushin's terms).

This representation in terms of β and γ is the most convenient to use in calculations, since nowhere on the surface do lines of constant values of the two parameters become parallel, unlike representations involving α.

The same equations apply in all four regions identified by Ilyushin. The line of symmetry corresponds to the line $\beta = 0$, while the boundaries between the "in-plane dominant" and "bending dominant" regions are given by $\beta = \pm 0.5$. The extreme edge of the surface is given by $\gamma = 0$.

NORMAL DIRECTION TO THE ILYUSHIN YIELD SURFACE

To make full use of the surface in its new parametrized form, it must be possible to define the direction of the normal to the surface.

If the yield surface is defined in terms of a yield function F such that

$$F(Q_t, Q_{tm}, Q_m) = 0. \tag{32}$$

By considering infinitesimally adjacent points on the yield surface, a tangential plane can be defined in terms of the generating parameters β and γ. The normal direction to this plane is found, and after some manipulation can be expressed as

$$F_t = \frac{\partial F}{\partial Q_t} = C(16K_2),$$
$$F_{tm} = \frac{\partial F}{\partial Q_{tm}} = C(-8K_1),$$
$$F_m = \frac{\partial F}{\partial Q_m} = C(K_0), \tag{33}$$

where C is a constant defining the magnitude of the normal vector.

The surface has been defined using an incremental theory of plasticity, in which the elastic strain resultant increments are negligible by comparison with the plastic components. Equations (19) thus represent expressions for the stress resultants in terms of the strain resultant increments; in the steady state, these must be the plastic strain resultant increments.

The J_i in eqn (19) are related to the K_i by eqn (25). With this substitution the resulting 6×6 matrix can be inverted analytically to give an expression for the plastic strain increment vector in terms of the loads applied to the plate:

$$\begin{bmatrix} d\varepsilon_x \\ d\varepsilon_y \\ d\varepsilon_{xy} \\ d\kappa_x \\ d\kappa_y \\ d\kappa_{xy} \end{bmatrix} = \frac{\sqrt{3P_\kappa}}{48(K_0K_2 - K_1^2)} \begin{bmatrix} 32K_2 & -16K_2 & 0 & -8K_1 & 4K_1 & 0 \\ -16K_2 & 32K_2 & 0 & 4K_1 & -8K_1 & 0 \\ 0 & 0 & 96K_2 & 0 & 0 & -24K_1 \\ -8K_1 & 4K_1 & 0 & 2K_0 & -K_0 & 0 \\ 4K_1 & -8K_1 & 0 & -K_0 & 2K_0 & 0 \\ 0 & 0 & -24K_1 & 0 & 0 & 6K_0 \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_{xy} \\ m_x \\ m_y \\ m_{xy} \end{bmatrix}. \tag{34}$$

If the K_i integrals in eqn (34) are replaced by the appropriate derivatives of F from eqns (33), then it may be seen that the right-hand side of eqn (34) is the normal vector to the Ilyushin yield surface in stress resultant space:

$$\begin{bmatrix} d\varepsilon_x \\ d\varepsilon_y \\ d\varepsilon_{xy} \\ d\kappa_x \\ d\kappa_y \\ d\kappa_{xy} \end{bmatrix} = \begin{bmatrix} \partial F/\partial n_x \\ \partial F/\partial n_y \\ \partial F/\partial n_{xy} \\ \partial F/\partial m_x \\ \partial F/\partial m_y \\ \partial F/\partial m_{xy} \end{bmatrix}, \tag{35}$$

where

$$\lambda = \frac{\sqrt{3}\cdot P_\kappa}{48C(K_0K_2-K_1^2)} \tag{36}$$

may be shown to be a positive non-zero quantity. Thus, the normality law is shown to hold in stress resultant space.

DESCRIPTION OF THE EXACT YIELD SURFACE

The yield surface is symmetrical about the line $Q_{tm} = 0$, and bounded by the edges given by the Schwarz inequality $Q_tQ_m \leqslant Q_{tm}^2$. It contains a discontinuity in slope at the point $Q_t = 1$, $Q_m = 0$, $Q_{tm} = 0$, but is smooth elsewhere. Three particular lines can be identified and are worth further discussion.

(i) *Boundary curve* ($Q_tQ_m = Q_{tm}^2$)

The boundary curve is given by the line $\gamma = 0$, and substituting this value into eqns (30) gives

$$\begin{aligned} K_0 &= \infty, \\ K_1 &= -2\beta+\beta K_0, \\ K_2 &= 0.25-3\beta^2+\beta^2K_0. \end{aligned} \tag{37}$$

By putting these values into eqns (29), the infinite values for K_0 cancel and simple parametric expressions for the boundary curve can be obtained:

$$\begin{aligned} Q_t &= 4\beta^2, \\ Q_{tm} &= -2\beta(1-4\beta^2), \\ Q_m &= (1-4\beta^2)^2. \end{aligned} \tag{38}$$

These equations are subject to the limit $-0.5 \leqslant \beta \leqslant 0.5$. The maximum value of Q_{tm} occurs when $\beta = \pm 1/\sqrt{12}$, whence $Q_t = 1/3$ ($=0.333333$), $Q_{tm} = \pm 2/3\sqrt{3}$ ($=0.384900$) and $Q_m = 4/9$ ($=0.444444$).

The normal to the surface along the boundary has to be found from eqns (33), taking account of the values of K_i derived above. K_0 is infinite, so the expressions for the other K_i become dominated by the K_0 term, to give

$$K_0 = \infty, \quad K_1 = \beta K_0, \quad K_2 = \beta^2 K_0. \tag{39}$$

The infinities then become included in the constant multiplier, to give

$$F_t = C \cdot 16\beta^2, \quad F_{tm} = -C \cdot 8\beta, \quad F_m = C \cdot 1. \tag{40}$$

(ii) *Line of symmetry* ($Q_{tm} = 0$)

Along the curve of symmetry $\beta = 0$ and the curve is defined by

$$Q_t = \gamma K_0^2, \quad Q_{tm} = 0, \quad Q_m = 16K_2^2, \tag{41}$$

where

$$K_0 = \log_e \left| \frac{\sqrt{0.25+\gamma}+0.5}{\sqrt{0.25+\gamma}-0.5} \right|,$$
$$K_2 = 0.5(\sqrt{0.25+\gamma}-\gamma K_0). \tag{42}$$

The two end points of the symmetry line are important. When $\gamma = 0$, which corresponds to pure bending of the plate,

$$Q_t = 0, \quad Q_{tm} = 0, \quad Q_m = 1,$$
$$F_t = 0, \quad F_{tm} = 0, \quad F_m = 1. \tag{43}$$

These are the same as the values obtained by putting $\beta = 0$ into the expression for the normals to the edge line [eqn (40)] indicating that there is no discontinuity at this extreme edge of the yield surface.

However, when $\gamma = \infty$, where the edge lines again meet the line of symmetry at the point corresponding to purely axial loading,

$$Q_t = 1, \quad Q_{tm} = 0, \quad Q_m = 0,$$
$$F_t = 0.8, \quad F_{tm} = 0, \quad F_m = 0.6. \tag{44}$$

These values for the normal direction differ from those obtained by putting $\beta = \pm 0.5$ into the equivalent expressions for the edge lines, indicating that there is a discontinuity of slope at this limiting point.

(iii) *The lines* $\beta = \pm 0.5$

The lines corresponding to $\beta = \pm 0.5$ are very significant in Ilyushin's original derivation, since they represent the boundary between the "in-plane dominant" and "bending dominant" regions; different equations were used to define the surface in the two regions. In the present formulation, these lines do not have special significance, other than representing the first of the values of β that form closed loops on the surface.

Physically, β represents the position within the thickness of the plate (given by the value of z) at which the equivalent strain increment intensity is a minimum. When the magnitude of β exceeds 0.5, there is no minimum within the section, so that in-plane forces predominate and overcome the tendency of the bending moments to cause a reversal of stress between the top and bottom surfaces.

DETERMINATION OF GENERAL POINTS ON THE YIELD SURFACE

The object of the reparametrization of the Ilyushin Yield surface is to allow the use of the exact surface in structural calculations. Thus, it must be demonstrated that the position of points on the surface can be calculated, as can the normal direction at those points.

Equations (29) and (30) define the full yield surface in terms of the new parameters β and γ. These are a set of non-linear equations which are not amenable to analytical solution; instead, a Newton–Raphson numerical procedure is adopted.

A number of potential problems could be tackled; for example, fixing two of the Q_i and calculating the third, but the most general question that will arise when solving plate problems is to take a set of known stress resultants and find the multiplier needed to put all the stress resultants onto the yield surface.

Thus, consider some point Q_1, Q_{12}, Q_2 in Q space that does not lie on the yield surface; it is desired to find the values β, γ and η for the corresponding point on the yield surface, such that

$$
\begin{aligned}
Q_t(\beta,\gamma) &= \eta Q_1, \\
Q_{tm}(\beta,\gamma) &= \eta Q_{12}, \\
Q_m(\beta,\gamma) &= \eta Q_2.
\end{aligned}
\tag{45}
$$

If estimates of β, γ and η are known, changes of these quantities can be found, using Newton's method, from

$$
\begin{bmatrix}
\dfrac{\partial Q_t}{\partial \beta} & \dfrac{\partial Q_t}{\partial \gamma} & Q_1 \\
\dfrac{\partial Q_{tm}}{\partial \beta} & \dfrac{\partial Q_{tm}}{\partial \gamma} & Q_{12} \\
\dfrac{\partial Q_m}{\partial \beta} & \dfrac{\partial Q_m}{\partial \gamma} & Q_2
\end{bmatrix}
\begin{bmatrix}
\Delta\beta \\ \Delta\gamma \\ -\Delta\eta
\end{bmatrix}
=
\begin{bmatrix}
\eta Q_1 - Q_t \\ \eta_{12} - Q_{tm} \\ \eta Q_2 - Q_m
\end{bmatrix}.
\tag{46}
$$

Convergence will have occurred when the right-hand side of these equations becomes tolerably small.

The rate of convergence for such problems depends on both the degree of conditioning of these equations, and the accuracy of the initial estimate of the solution. As can be seen from Figs 7 and 8, the parameters β and γ are almost orthogonal for values of $Q_t - Q_m$ close to -1, but as $Q_t - Q_m$ increases, the angle between the β and γ curves decreases until they become almost parallel near the singularity point ($Q_t = 1$, $Q_m = 0$, $Q_{tm} = 0$), and hence ill-conditioned. Nevertheless, after studying plots of all combinations of α, β and γ, it was decided that the $\beta - \gamma$ combination offered the best conditioning over most of the region, and it has been found possible in practice to determine points on the surface using this combination right up to the singularity point (although with considerably more iterations for the ill-conditioned region).

The choice of starting point is almost always critical when using Newton's method, and is particularly so when dealing with ill-conditioned equations. Furthermore, since any solution routine that is used in practice is likely to be buried deep within a finite element program, an initial estimate is required that can be derived *ab initio* and does not depend on the values obtained in a previous solution.

An efficient and accurate starting value routine can be established by making use of the properties of the Ilyushin yield surface and the Ivanov approximation to it. Equations (33) can be used to relate the yield surface parameters to the direction of the normal to the yield surface. It is then possible to find the normal direction to the Ivanov approximate yield surface at the point in question and use this to find estimates of the starting parameters for the Newton–Raphson iteration.

Thus, by substituting eqns (33) into eqn (24), a relationship between the normals to the surface and the parameters α and β is obtained:

$$\begin{bmatrix} Q_t \\ Q_{tm} \\ Q_m \end{bmatrix} = \eta \begin{bmatrix} Q_1 \\ Q_{12} \\ Q_2 \end{bmatrix} = C_1^2 \begin{bmatrix} F_m^2 & -F_{tm}F_m/8 & F_{tm}^2/64 \\ -F_{tm}F_m/2 & (4F_{tm}F_m+F_{tm}^2)/32 & -F_tF_m/32 \\ F_{tm}^2/4 & -F_tF_{tm}/8 & F_t^2/16 \end{bmatrix} \begin{bmatrix} \alpha \\ -2\beta \\ 1 \end{bmatrix}, \quad (47)$$

where C_1 is an arbitrary constant.

Ivanov's approximate yield function [eqn (13)] can be differentiated to give approximate values for the F_i derivatives:

$$F_t = \frac{\partial F}{\partial Q_1} = C_2\left(1 - \frac{(0.12Q_2^2+Q_{12}^2/4)}{(Q_1+0.48Q_2)^2}\right),$$

$$F_{tm} = \frac{\partial F}{\partial Q_{12}} = C_2\left(\frac{Q_{12}}{\sqrt{Q_2^2/4+Q_{12}^2}} + \frac{Q_{12}/2}{(Q_1+0.48Q_2)}\right),$$

$$F_m = \frac{\partial F}{\partial Q_2} = C_2\left(0.5 + \frac{Q_2/4}{\sqrt{Q_2^2/4+Q_{12}^2}} - \frac{(Q_1^2/4+0.12Q_{12}^2)}{(Q_1+0.48Q_2)^2}\right), \quad (48)$$

where C_2 is another arbitrary constant.

Ivanov's approximation is of the form

$$F(Q_t, Q_{tm}, Q_m) = 1, \quad (49)$$

and, since it is of order 1 in terms of the Q_i, it follows that

$$F(\eta Q_1, \eta Q_{12}, \eta Q_2) = \eta F(Q_1, Q_{12}, Q_2) = 1. \quad (50)$$

It is also known (Robinson, 1971) that Ivanov's yield surface is within 1% of the true value, so a reasonable starting value for η is given by

$$\eta_0 = \frac{1}{Q_1+Q_2/2+\sqrt{Q_2^2/4+Q_{12}^2}-\frac{1}{4}\left(\frac{Q_1Q_2-Q_{12}^2}{Q_1+0.48Q_2}\right)}. \quad (51)$$

Equations (47) now give three linear equations in the three unknowns α, β and C $(=C_1C_2^2/\eta)$; all other terms may be expressed in terms of the known quantities Q_1, Q_{12} and Q_2. They may be further reduced to two equations in α and β, since the value of C is not needed, which can then be solved for the initial values α_0, β_0 and hence γ_0.

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix}, \qquad \gamma_0 = \alpha_0 - \beta_0^2 \quad (52)$$

where

$$a_{11} = 64Q_2\left(\frac{\partial F}{\partial Q_2}\right)^2 - 16Q_1\left(\frac{\partial F}{\partial Q_{12}}\right)^2,$$

$$a_{12} = 16Q_2\frac{\partial F}{\partial Q_{12}}\cdot\frac{\partial F}{\partial Q_2} - 16Q_1\frac{\partial F}{\partial Q_1}\cdot\frac{\partial F}{\partial Q_{12}},$$

$$a_{21} = 32Q_2\frac{\partial F}{\partial Q_{12}}\cdot\frac{\partial F}{\partial Q_2}+16Q_{12}\left(\frac{\partial F}{\partial Q_{12}}\right)^2,$$

$$a_{22} = 4Q_2\left(4\frac{\partial F}{\partial Q_1}\cdot\frac{\partial F}{\partial Q_2}+\left(\frac{\partial F}{\partial Q_{12}}\right)^2\right)+16Q_{12}\frac{\partial F}{\partial Q_1}\cdot\frac{\partial F}{\partial Q_{12}}, \tag{53}$$

and

$$b_1 = -Q_2\left(\frac{\partial F}{\partial Q_{12}}\right)^2+4Q_1\left(\frac{\partial F}{\partial Q_1}\right)^2,$$

$$b_2 = -2Q_2\frac{\partial F}{\partial Q_1}\cdot\frac{\partial F}{\partial Q_{12}}-4Q_{12}\left(\frac{\partial F}{\partial Q_1}\right)^2. \tag{54}$$

This algorithm predicts the correct sign of β_0 and a value of $\beta_0 = 0$ when $Q_{12} = 0$. It also correctly predicts that $\gamma_0 = 0$ for $Q_1Q_2 = Q_{12}^2$ and the correct value of β along this boundary. At the singular point $(1, 0, 0)$, the Ivanov normals cannot be calculated, so no values are forthcoming for this case; this must be treated as a special case in any computer application of the method.

The method requires knowledge [in eqns (46)] of the derivatives of the Q_i with respect to β and γ. In areas where the equations are well conditioned, these are often obtained sufficiently accurately, and more conveniently, by numerical differentiation. However, they are often required to be more accurate, and can be obtained analytically from the chain rule.

To summarize the procedure:

(1) for a known set of (Q_1, Q_{12}, Q_2);
(2) find derivatives to Ivanov's yield surface from (33);
(3) find η_0 from (35);
(4) find α_0, β_0 and γ_0 from (36), (37) and (38);
(5) differentiate (16) and (17) with respect to β and γ;
(6) substitute into (31), and solve for changes in β, γ and η;
(7) repeat (5) and (6) until convergence occurs. In well conditioned regions, it is usually unnecessary to refine the estimates of the derivatives in (5), so this step only needs to be carried out once.

This procedure will converge for all points on the exact yield surface, except for points immediately adjacent to the singularity point $(1, 0, 0)$. In most cases, only two iterations are required, but for values of $Q_t - Q_m$ approximately 0.9999, about 15 iterations may be required, and it will probably be necessary to employ a computer model that uses double precision for the inner loop of the Newton–Raphson iterations.

Since this process converges so quickly, it can be used as the core of other procedures. For example, the lines of constant Q_i drawn on Fig. 1 were obtained by specifying two of (Q_t, Q_{tm}, Q_m) and allowing the third to vary. An outer Newton–Raphson loop was constructed which varied the starting values of (Q_1, Q_{12}, Q_2) until the resulting points on the yield surface had the required fixed values.

VISUALIZATION OF NORMAL TO YIELD SURFACE

It was shown above that there is a discontinuity in the slope of the yield surface when the plate is subject to purely in-plane forces. Although there is no discontinuity elsewhere, there is a "lip" on the surface along the edge $Q_tQ_m = Q_{tm}^2$.

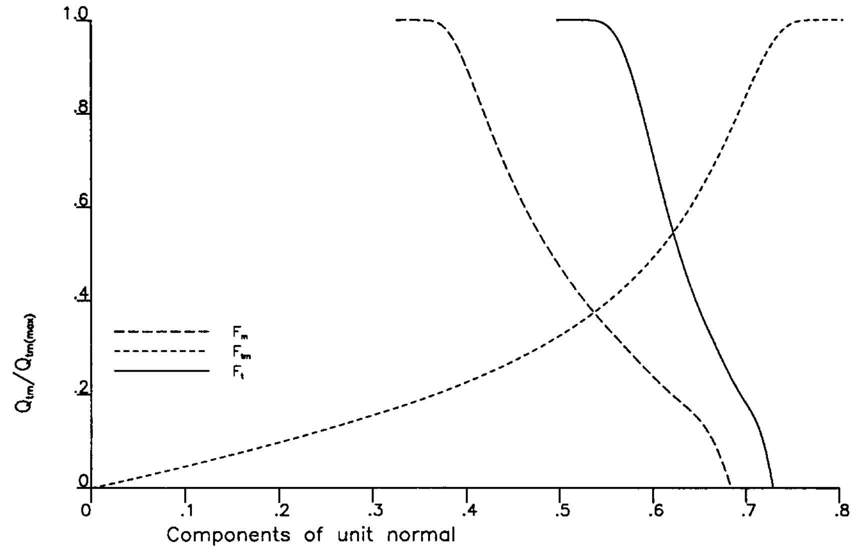

Fig. 9. Components of normal to yield surface along $Q_t = Q_m$.

Figure 9 shows the three components of the unit normal to the yield surface along a line on the surface corresponding to $Q_t = Q_m$, with Q_{tm} varying from 0 to the maximum value at the edge ($Q_{tm(\max)}$). The components on the edge are given by eqns (40), which rely on γ being zero, and hence K_0 being infinite. Elsewhere, the components of the normal are given by eqns (33), using values of K_i from eqns (30). Although these equations give the same values on the edge, it is clear from the figure that there is a sudden change in the direction to the normal immediately adjacent to the edge of the surface. It is possible to follow this sudden change in the normal direction using eqns (30), but special procedures have to be adopted which allow the denominator in the expression for K_0 to be very small. This can be done by assuming γ is very small and using the binomial expansion to simplify the calculation. This leads to

$$K_0 = \log_e(1-4\beta^2) - \log_e(\gamma), \tag{55}$$

which can be evaluated provided logarithms of small numbers can be calculated.

The corollary of this observation is that there must be a small lip on the yield surface at the edge. It is not possible to see this lip on the surface, unless sections are taken and magnified by several orders of magnitude. The lip is present all along the edge, except at the pure bending point, where the normal on the edge and the normal to the line of symmetry coincide, as discussed above. Figure 10 shows a three-dimensional view of the surface; the short straight lines represent normals to the surface. These are shown along radial sections which all pass through the $Q_t = 0$, $Q_m = 0$ axis. Also shown are normals to the edge line and normals to the line of $\gamma = 0.00001$, which is effectively immediately adjacent to the edge line. The true discontinuity in the surface occurs for purely in-plane loading.

This observation is of some significance, since beams and axes of symmetry in plates, are usually loaded in such a way that they would lie on this edge of the yield surface. Calculations of the rigidities may become significant, especially in stability problems, where small changes in stiffness will lead to large changes in the buckling load.

NUMERICAL EXAMPLE

In the companion paper (Burgoyne and Brennan, 1992), a numerical example will be given showing the process of deriving a position on the yield surface; this will be done in association with the method for deriving accurate elasto-plastic rigidities for a plate from the normal direction of the exact Ilyushin yield surface.

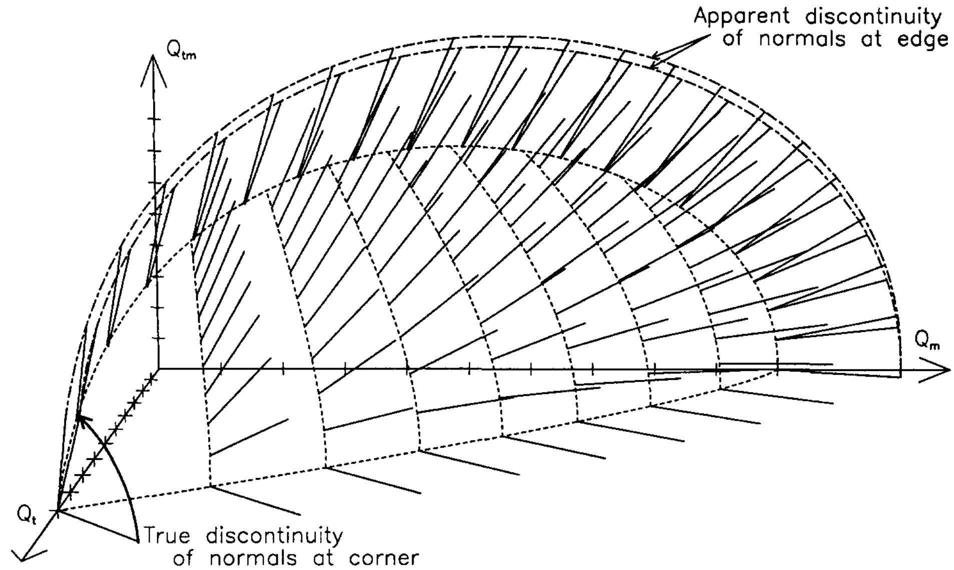

Fig. 10. Three-dimensional view showing normals to yield surface, discontinuity at one corner and lip at the edge.

CONCLUSIONS

A method has been presented for the reparametrization of the Exact Ilyushin Yield Surface. Detailed equations have been given from which the surface can be constructed, and it has been shown that it is practical to determine positions on this yield surface corresponding to a proportional increase of any given set of stress resultants, by a simple iterative procedure. Reliable starting values for this iteration can be found which allow convergence to take place in 2 or 3 iterations from most initial positions.

It has been shown that the normal direction to this surface can be calculated, and in a companion paper these values will be used to determine the corresponding elasto-plastic tangential rigidity matrix.

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